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Exact solution for light propagation through inhomogeneous media

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Abstract : In inhomogeneous media, the index of refraction is a coordinate-dependent parameter. In this domain, the exact analytical solutions for the electric and magnetic fields can not in general, be found and numerical methods must be used. In this paper, a suitable algorithm, using the supersymmetric quantum mechanical approach has been given for the transmission of light through some special cases of these media which have spatial one-dimensional coordinate-dependent index of refractions (shape-invariant index of refractions) Exact solutions for electromagnetic fields in these media are expressed in terms of well known special functions

Keywords : Light propagation, exact solution, inhomogeneous media

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1. Introduction

Numerous advances have been made in the electromagnetic theory in recent years. This is, in part, to new applications of the theory to many practical problems. For example, in microwave and millimeter wave applications, there is an interesting need to investigate the electromagnetic problems of new guiding structures, phase array, microwave imaging, polarimetric radars, microwave hazards, frequency-sensitive surfaces, composite materials, and microwave remote sensing. In the field of optics, applications involving fibre optics, integrated optics, atmospheric optics, light diffusion in tissues, and optics for inhomogeneous media are among many problems whose solutions require the use of electromagnetic theory as an essential element [1,2]. Guided waves in inhomogeneous planar optical waveguide, have received considerable attention owing to their ability to perform optical signal processing and optical computing. For this purpose, the characteristic of above mentioned waveguide with a Kerr-like film layer have also been extensively investigated for ultra-high speed optical processing [1,2].

Some published papers so far, are based on numerical methods [3]. In an analytical approach, the supersymmetry method for index of refraction in special case such as $n^2(x) = a - bx^2$, where a and b are constant parameters, was discussed [4]. The interval of x using the constant parameters must be such that $n(x)$ is real and positive. In this paper using paraxial approximation, we will present the analytical approach for light transmission through inhomogeneous optical waveguide. For this purpose, we want to establish the relation between Schrödinger and Helmholtz equations from a supersymmetric point of view. Using the previous works about supersymmetry and shape invariant potentials in quantum mechanical studies [5,6], one can obtain the exact solution to electric and magnetic fields for shape invariant index of refractions which can occur in optical domains for inhomogeneous media in reality. Organization of this paper is as follows : In Section 2, the differential equation for the electromagnetic fields in the time harmonic cases are related to the Schrödinger equation. Then, by using the supersymmetric method, the exact solutions for shape invariant index of refractions are introduced and the electromagnetic fields in terms of well known special functions obtained. Finally, the paper ends with a conclusion.

2. Helmholtz equation and its solution

In this section, we will obtain the differential equation for y-component of the electric field in waveguide as shown in

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Figure 1. Let us consider a plane wave incident upon a medium whose dielectric constant is a function of height x.

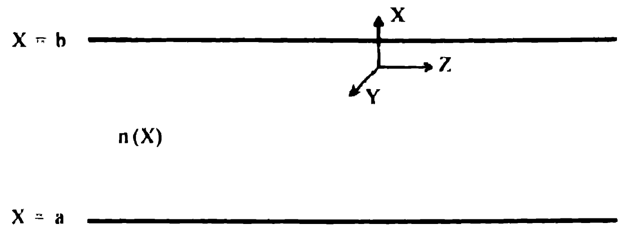

Figure 1. Schematic diagram of waveguide

We choose the z-axis specially such that the plane of incidence is in the $x-z$ plane. This is a two dimensional problem $\left(\frac{\partial}{\partial y}=0\right)$, and thus there are two independent, TE and TM waves. In the first case TE, we combine Maxwell's equations as

$$\begin{aligned} \nabla\times E &= -i\omega\mu H, \\ \nabla\times H &= -i\omega\varepsilon E, \end{aligned} \tag{1}$$

and get

$$\begin{aligned} \nabla\times\nabla\times E &= \nabla(\nabla . E)-\nabla^2 E \\ &= \omega^2\mu\varepsilon E. \end{aligned}$$

Noting that for dielectric media in the absence of any charge-density anywhere ($\nabla . E=0$) and $\frac{\partial E_y}{\partial y}=0$, we get

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\omega^2\mu\varepsilon(x)\right]E_y(x,z)=0, \tag{2}$$

where $\omega^2\mu\varepsilon(x)=k_0^2 n^2(x)$ and for time harmonic situations, $E_y(x,z,t)=E_y(x,z)e^{-i\omega t}$. For TM, we eliminate the electric field (E) from eq. (1) and get

$$\begin{aligned} \nabla\times E &= \nabla\times\frac{1}{i\omega\varepsilon(x)}\nabla\times H \\ &= -i\omega\mu H. \end{aligned}$$

Rewriting this using $H=H_y\hat{y}$ and $\frac{\partial}{\partial y}=0$, we get

$$\frac{\partial^2}{\partial z^2}H_y+\varepsilon(x)\frac{\partial}{\partial x}\left[\frac{1}{\varepsilon(x)}\frac{\partial}{\partial x}H_y\right]+\omega^2\mu\varepsilon(x)H_y=0. \tag{3}$$

We may simplify eq. (3) by a suitable change of H_y to U as

$$U=\frac{H_y}{n(x)}$$

and get

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k_0^2N^2(x)\right]U=0, \tag{4}$$

where $k_0^2N^2(x)=k_0^2n^2+\frac{n''}{n}-\frac{2n'^2}{n^2}$. Note that eqs. (2) and (4) have the same mathematical form. Thus, the study of the electromagnetic wave equation for the TE wave may be easily extended to that of the TM wave with changes from n to N. Let us now consider a TE plane wave obliquely incident upon the medium in Figure 1. We write

$$E_y(x,z)=E_y(x)e^{-ikz}, \tag{5}$$

then Helmholtz equation for light transmission in this waveguide is

$$\frac{d^2}{dx^2}E_y(x)+\left(k_0^2n^2(x)-k^2\right)E_y(x)=0 \tag{6}$$

and we want to solve exactly this equation for the given $n^2(x)$. For TM modes, we have the equation similar to eqs (2–6). From the physical point of view, there is a close analogy between quantum mechanics and Helmholtz optics. In the first case, the wave function describes the state function of particle with in the later case, the electromagnetic field represents the state function for photons. With this similarity, the index of refraction has the same role as a potential in Schrödinger equation. Hence, the same mathematical techniques can be applied for both cases. Here, we use the supersymmetric approach for our discussion. In supersymmetric quantum mechanics (SUSY), the SUSY partner Hamiltonians [5–7] are given by

$$H_{\pm}=-\frac{d^2}{dx^2}+V_{\pm}(x)$$

with $V_{\pm}=W^2(x)\pm W'(x)$,

where $W(x)$ is the solution of Riccati equation and named super-potential, as partner potentials. Also, in the practical cases, the potential depends on some constant parameters which determine the physical behaviour of systems. The key concept of shape-invariance (SI) was introduced by Gendenshtein [7] for partner potentials $V_{\pm}$ is as follows :

$$V_+(a_0,x)=V_-(a_1,x)+R(a_1),$$

where a_1 is a function of the constant parameter a_0 given by $a_1=F(a_0)$ and the residual term $R(a_1)$ is independent of the variable x. It is shown that for such potentials (SIP), energy and the wave functions can be obtained algebraically [5,6,8]. Because of two different partner potentials in the formulation of supersymmetric quantum mechanics, we must have two different indices of refraction ($n_{\pm}$) in Helmholtz problems as

$$n_{\pm}^2(x)=n_0^2+n_1^2f_{\pm}(x), \tag{7}$$

where n_0 and n_1 are constants and $f_{\pm}(x)$ are the coordinate-dependent part of the indices of refraction which are related to the partner potentials through the following relation :

$$V_{\pm}(x)=-k_0^2n_1^2f_{\pm}(x).$$

Also, the quantum mechanical energy ε and the wave vector k satisfy the following relation

$$\varepsilon=k_0^2n_0^2-k^2.$$

As an example, we investigate $E_y(x)$ for the case where the index of refraction satisfies $-k_0^2 n_1^2 f_-(x) = A^2 - \frac{A(A+\alpha)}{\cosh(\alpha x)^2}$ and for other cases, the results are given in Table 1. For this case, the Helmholtz equation converted to

$$\frac{d^2 E_y(x)}{dx^2} + \left[k_0^2 n_0^2 - k^2 - A^2 + \frac{A(A+\alpha)}{\cosh^2(\alpha x)}\right] E_y(x) = 0.$$

Using [5,6], the $E_y(x)$ and k can be obtained as follows :

$$E_y(x,z,t) = e^{i(\omega t - kz)}(\cosh(\alpha x))^{-s} P_n^{-s-\frac{1}{2},-s-\frac{1}{2}}(i\sinh(\alpha x)),$$

$$k^2 = k_0^2 n_0^2 - A^2 + (A - n\alpha)^2,$$

where n is a positive integer, $A \geq 0$, $s = \frac{A}{\alpha}$ and $P_n^{\alpha,\beta}(x)$ are the Jacobi polynomials [9].

Table 1. Some of the solvable indices of refraction and their response $\hbar = 1,\ 2m = 1,\ A,\ B,\ \alpha,\ \omega,\ l > 0$

N	$-k_0^2 n_1^2 f(x)$		$E_y(x)$	
	$\frac{1}{4}\omega^2\left(x - \frac{2b}{\omega}\right)^2 - \omega/2$ $-\infty < x < +\infty$	$y = \sqrt{\frac{\omega}{2}}\left(x - \frac{2b}{\omega}\right)$	$e^{-\frac{y^2}{2}} H_n(y)$	$\sqrt{k_0^2 n_0^2 - i}$
2	$\frac{1}{4}\omega^2 x^2 + \frac{l(l+1)}{x^2} - (l+3/2)\omega$ $0 < x < \infty$	$\frac{1}{2}\omega x^2$	$y^{\frac{l+1}{2}} e^{-\frac{1}{2}y} L_n^{l+\frac{1}{2}}(y)$	$\sqrt{k_0^2 n_0^2 - 2n\omega}$
3	$-\frac{e^2}{x} + \frac{l(l+1)}{x^2} + \frac{e^4}{4(l+1)^2}$ $0 < x < \infty$	$\frac{xe^2}{(n+l+1)}$	$y^{l+1} e^{-\frac{1}{2}y} L_n^{2l+1}(y)$	$\sqrt{k_0^2 n_0^2 - Q}$ $Q = \frac{e^4}{4}\left(\frac{1}{(l+1)^2} - \frac{1}{(n+l+1)^2}\right)$
4	$A^2 + B^2 e^{(-2\alpha)x} - 2B(A+\alpha/2)e^{-\alpha x}$ $-\infty < x < +\infty$	$(2B/\alpha)e^{-\alpha x}$ $s = A/\alpha$	$y^{s-n} e^{-\frac{1}{2}y} L_n^{2s-2n}(y)$	$\sqrt{k_0^2 n_0^2 - A^2 + (A - n\alpha)^2}$
5	$A^2 + (B^2 - A^2 - A\alpha)\sec h^2(\alpha x)$ $+ B(2A+\alpha)\sec h(\alpha x)\tan h(\alpha x)$ $-\infty < x < +\infty$	$\sin h(\alpha x)$ $s = A/\alpha$ $\lambda = B/\alpha$	$i^n (1+y^2)^{-s/2} e^{-\lambda \tan^{-1}(y)}$ $\times P_n^{(i\lambda - s - 1/2, -i\lambda - s - 1/2)}(iy)$	$\sqrt{k_0^2 n_0^2 - A^2 + (A - n\alpha)^2}$
6	$A^2 + B^2/A^2 - A(A+\alpha)\sec h^2(\alpha x)$ $+ 2B\tan h(\alpha x)$ $-\infty < x < +\infty$	$\tan h(\alpha x)$ $s = A/\alpha,\ \lambda = B/\alpha^2$ $a = \lambda/(s-n)$ $B < A^2$	$(1-y)^{(s-n+a)/2}(1+y)^{(s-n-a)/2}$ $\times P_n^{(s-n+a, s-n-a)}(y)$	$\sqrt{k_0^2 n_0^2 - A^2 - B^2/A^2 + Q}$ $Q = (A - n\alpha)^2 + B^2/(A - n\alpha)^2$
7	$A^2 + B^2/A^2 + A(A+\alpha)\mathrm{cosec}\,h^2(\alpha x)$ $-2B\cot\tan h(\alpha x)$ $0 < x < +\infty$	$\cot h(\alpha x)$ $s = A/\alpha,\ \lambda = B/\alpha^2$ $a = \lambda/(n+s)$ $B > A^2$	$(y-1)^{-(s+n-a)/2}(1+y)^{(s+n+a)/2}$ $\times P_n^{(-s-n+a, -s-n-a)}(y)$	$\sqrt{k_0^2 n_0^2 - A^2 - B^2/A^2 + Q}$ $Q = (A + n\alpha)^2 + B^2/(A + n\alpha)^2$
8	$-A^2 + (A^2 + B^2 - A\alpha)\sec^2(\alpha x)$ $-B(2A-\alpha)\tan(\alpha x)\sec(\alpha x)$ $-\frac{4(1)}{2} < \alpha x < +\frac{4(1)}{2}$	$\sin(\alpha x)$ $s = A/\alpha,\ \lambda = B/\alpha$	$(1-y)^{(s-\lambda)/2}(1+y)^{(s+\lambda)/2}$ $\times P_n^{(s-\lambda-1/2, s+\lambda-1/2)}(y)$	$\sqrt{k_0^2 n_0^2 + A^2 - (A + n\alpha)^2}$
9	$A^2 + (A^2 + B^2 + A\alpha)\mathrm{cosec}\,h^2(\alpha x)$ $-B(2A+\alpha)\cot h(\alpha x)\mathrm{cosec}\,h(\alpha x)$ $0 < x < +\infty$	$\cos h(\alpha x)$ $s = A/\alpha,\ \lambda = B/\alpha$	$(y-1)^{(\lambda-s)/2}(1+y)^{-(s+\lambda)/2}$ $\times P_n^{(\lambda-s-1/2, -s-\lambda-1/2)}(y)$	$\sqrt{k_0^2 n_0^2 - A^2 + (A - n\alpha)^2}$
10	$-A^2 + B^2/A^2 + A(A-\alpha)\mathrm{cosec}^2(\alpha x)$ $+2B\cot(\alpha x)$ $-\frac{1}{2} < x < +\frac{1}{2}$	$i\cot(\alpha x),\ s = A/\alpha$ $\lambda = B/\alpha,\ a = \lambda/(s+n)$	$(y^2-1)^{-(s+\lambda)/2,(a\alpha x)}$ $\times P_n^{(-s-\lambda+ia, -s-\lambda-ai)}(y)$	$\sqrt{k_0^2 n_0^2 + A^2 - (A + n\alpha)^2 + Q}$ $Q = B^2/(A + n\alpha)^2 - B^2/A^2$

As noted in introduction for $n^2(x) = a - bx^2$, the interval of x for results given in Table 1 are restricted cases and in practice, must be determined for the given n_0, n_1 and k_0 parameters such than $n(x)$ is real and positive.

The y-component of the electric fields for some special cases are shown in the Figures (2–5)

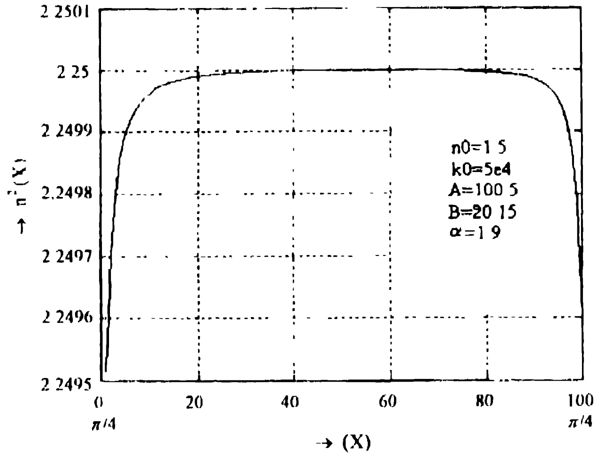

Figure 2. The index of refraction profile

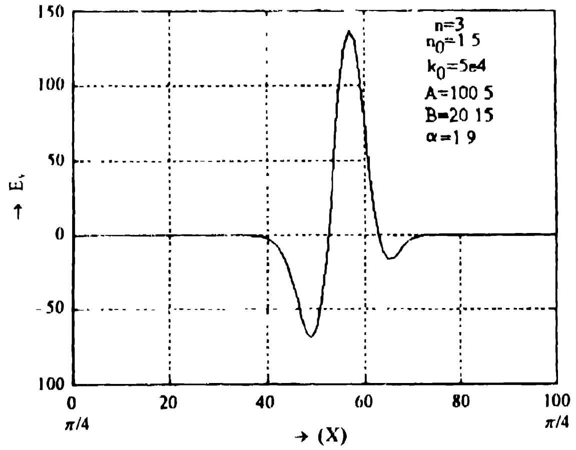

Figure 3. E_y as a function of x

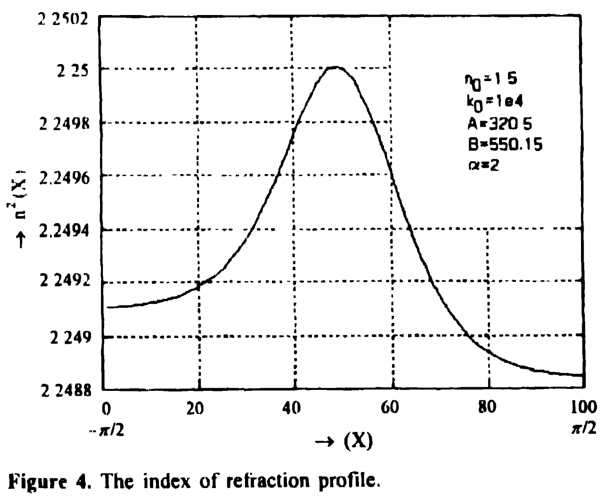

Figure 4. The index of refraction profile.

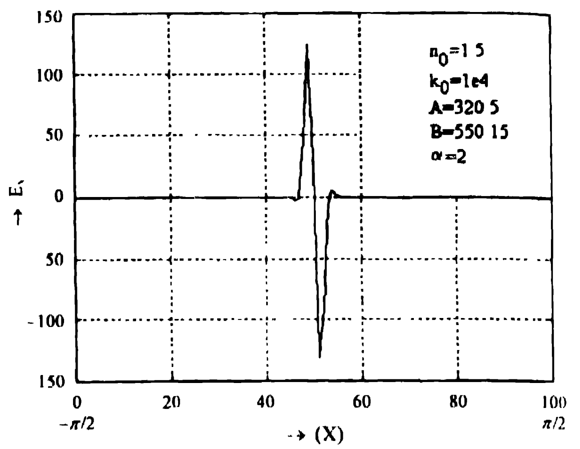

Figure 5. E_y as a function of x

3. Conclusion

In this paper, the relation between the Schrödinger and a special (1D) case of the Maxwell equations is clearly explained and the electromagnetic field as a photon wave function [10] is obtained from supersymmetric quantum mechanical methods. In this work, the wave vector in z-direction (k) is related to the eigenvalues, and is quantized That is, only the special modes can be transmitted. Also a list of indices of refraction as nonlinear functions of x having exact solutions, is introduced. Though these indices of refraction are one-dimensional functions, this approach can be applied in higher dimensional cases also. The practically realizable indices of refraction and electric fields are displayed against x-coordinates in Figures (2, 3) and in Figures (4, 5) respectively. As an example, a quarter wave multilayer stack can be used for the implementation of the proposed index of refraction. These indices of refraction which have slow variation with x, can be approximated by constant indices of refraction over short intervals of x, similar to the technique of Simpson integration method. The refractive index curve can be displayed with rectangular functions displaced from each other with amplitudes dependent on x. These structures may be realized by using molecular beam epitaxy (MBE) technology in solid state electronics.

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